EXERCISES [MAI 2.7-2.8]

COMPOSITION – INVERSE FUNCTIONS

SOLUTIONS

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A. Paper 1 questions (SHORT)

1. (a) $(f \circ g)(x) = 10 - 10x$ $\quad (g \circ f)(x) = 50 - 10x$

(b) $f^{-1}(x) = (10 - x)/2$,

(c) $g^{-1}(10) = 2$

(d) $(f^{-1} \circ g)(x) = (10 - 5x)/2$, $\quad (g \circ f)^{-1}(x) = (50 - x)/10$

(e) $(f \circ f)(x) = 4x - 10$ $\quad (g \circ g)(x) = 25x$

2. (a) (i) $(f \circ g)(x) = 2(5x+3)+5 = 10x+11$, then $(f \circ g)(1) = 21$

(ii) $(f \circ g)(1) = f(g(1)) = f(8) = 21$

(b) (i) $f(x) = y \Leftrightarrow 2x+5 = y \Leftrightarrow x = (y-5)/2$ so $f^{-1}(x) = (x-5)/2$, then $f^{-1}(25) = 10$

(ii) $f(x) = 25 \Leftrightarrow 2x+5 = 25 \Leftrightarrow x = 10$, so $f^{-1}(25) = 10$

(c) $(g \circ f)(1) = g(f(1)) = g(7) = 38$

$g(x) = 53 \Leftrightarrow 5x+3 = 53 \Leftrightarrow x = 10$, so $g^{-1}(53) = 10$.

3. (a) $g(3) = 1$ $\quad f^{-1}(3) = 4$

(b) $(f \circ g)(2) = -1$

(c) $(g \circ g)(3) = 5$

(d) $x = 1$

4. (a) $(g^{-1} \circ f)(4) = 3$

(b) $x = 1$

(c) $(g^{-1} \circ g)(2) = 2$

5. (a) $2x+1 = y \Leftrightarrow x = \dfrac{y-1}{2} \Leftrightarrow f^{-1}(x) = \dfrac{x-1}{2}$

(b) $g(f(-2)) = g(-3) = 3(-3)^2 - 4 = 23$

(c) $f(g(x)) = f(3x^2-4) = 2(3x^2-4)+1 = 6x^2-7$

6. $\sqrt{3-2x} = 5 \Leftrightarrow 3 - 2x = 25 \Leftrightarrow -2x = 22 \Leftrightarrow x = -11$

OR

Let $y = \sqrt{3-2x} \Rightarrow y^2 = 3 - 2x \Rightarrow x = \dfrac{3-y^2}{2} \Rightarrow f^{-1}(x) = \dfrac{3-x^2}{2}$

$\Rightarrow f^{-1}(5) = \dfrac{3-25}{2} = -11$

7. (a) $(h \circ g)(x) = \frac{5(3x-2)}{(3x-2)-4} = \frac{5(3x-2)}{(3x-6)} = \frac{15x-10}{3x-6}$

(b) $x = \frac{2}{3}$ $(= 0.667)$

8. (a) $(f \circ g): x \mapsto 3(x + 2) = 3x + 6$

(b) **METHOD 1**

$f^{-1}(x) = \frac{x}{3}$ $g^{-1}(x) = x - 2$

$f^{-1}(18) + g^{-1}(18) = 6 + 16 = 22$

METHOD 2

$3x = 18, x + 2 = 18$

$x = 6, x = 16$

$f^{-1}(18) + g^{-1}(18) = 6 + 16 = 22$

9. (a) $g^{-1}(x) = \frac{x+3}{2}$

(b) **METHOD 1**

$g(4) = 5$, $f(5) = 25$

METHOD 2

$(f \circ g)(x) = (2x - 3)^2$

$(f \circ g)(4) = (2 \times 4 - 3)^2 = 25$

10. (a) $(g \circ f)(x) = 7 - 2x + 3 = 10 - 2x$

(b) $g^{-1}(x) = x - 3$

(c) **METHOD 1**

$g^{-1}(5) = 2$, $f(2) = 3$

METHOD 2

$(f \circ g^{-1})(x) = 7 - 2(x - 3), 13 - 2x$ so $(f \circ g^{-1})(5) = 3$

11. (a) **METHOD 1**

$f(3) = \sqrt{7}$ $(g \circ f)(3) = 7$

METHOD 2

$(g \circ f)(x) = \sqrt{x+4}^2 = x + 4$

$(g \circ f)(3) = 7$

(b) $f^{-1}(x) = x^2 - 4$

(c) $x \geq 0$

12. (a) **METHOD 1**

For $f(-2) = -12$

$(g \circ f)(-2) = g(-12) = -24$

METHOD 2

$(g \circ f)(x) = 2x^3 - 8$

$(g \circ f)(-2) = -24$

(b) $x^3 - 4 = y \Leftrightarrow x = \sqrt[3]{y+4}$ so $f^{-1}(x) = \sqrt[3]{(x+4)}$

13. (a) $3x + 5 = 2 \Leftrightarrow x = -1$

$f^{-1}(2) = -1$

(b) $g(f(-4)) = g(-12 + 5) = g(-7) = 2(1 + 7) = 16$

14. (a) $f(3) = 2^3$

$(g \circ f)(3) = \dfrac{2^3}{2^3 - 2} = \dfrac{8}{6} = \dfrac{4}{3}$

(b) $\dfrac{x}{x-2} = y \Leftrightarrow xy - 2y = x \Leftrightarrow x(y-1) = 2y \Leftrightarrow x = \dfrac{2y}{y-1}$ so $y = \dfrac{2x}{(x-1)}$

$y = \dfrac{10}{(5-1)} = 2.5$

15. (a) $\dfrac{x}{x+5} = y \Leftrightarrow x = xy + 5y \Leftrightarrow x - xy = 5y \Leftrightarrow x(1-y) = 5y \Leftrightarrow x = \dfrac{5y}{1-y}$

$f^{-1}(x) = \dfrac{5x}{1-x}$

(b) $R = \dfrac{5S}{1-S}$

16. (a) $(f \circ g)(x) = \dfrac{2}{x-3}$

(b) $(f \circ g)(4) = 2$

(c) $P = \dfrac{2}{S-3}$

(d) $P = 2$

17. Just confirm that $f^{-1}(x) = \dfrac{2x-3}{3x-2}$ and $(f \circ f)(x) = x$

18. (a) $y = \dfrac{6-x}{2} \Rightarrow x = 6 - 2y$

$\Rightarrow g^{-1}(x) = 6 - 2x$

(b) $(f \circ g^{-1})(x) = 4[(6 - 2x) - 1] = 4(5 - 2x) = 20 - 8x$

$20 - 8x = 4 \Rightarrow 8x = 16 \Rightarrow x = 2$

19. $(f \circ g) : x \mapsto x^3 + 1$

$(f \circ g)^{-1} : x \mapsto (x - 1)^{1/3}$

20. (a) $h(x) = \dfrac{4}{x+2} - 1 \quad \left(= \dfrac{2-x}{2+x} \right)$

(b) $\dfrac{4}{x+2} - 1 = y \Leftrightarrow \dfrac{4}{x+2} = y + 1 \Leftrightarrow \dfrac{x+2}{4} = \dfrac{1}{y+1} \Leftrightarrow x + 2 = \dfrac{4}{y+1} \Leftrightarrow x = \dfrac{4}{y+1} - 2$

(c) (i) $h(x)$ represents an expression of K in terms of M

(ii) $h^{-1}(x)$ represents an expression of M in terms of K

(iii) $h(3)$ represents the value K when $M = 3$

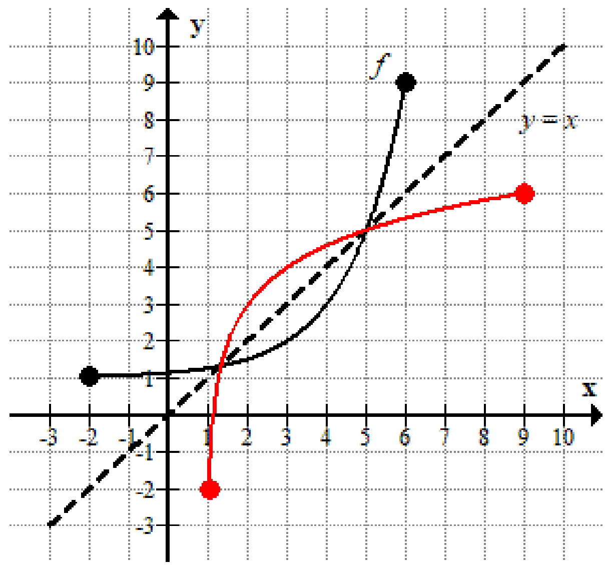
(iv) $h^{-1}(0.5)$ represents the value M when $K = 0.5$

21. $f^{-1}(x) = \dfrac{x+5}{3}$ $\quad g^{-1}(x) = x+2$

$(f^{-1} \circ g)(x) = \dfrac{x+3}{3}$ $\quad (g^{-1} \circ f)(x) = 3x - 3$

$\dfrac{x+3}{3} = 3x - 3 \ (x + 3 = 9x - 9) \ \Rightarrow x = \dfrac{12}{8} = \dfrac{3}{2}$

22.



23. (a) $f(x) = (x-3)^2 + 4 = x^2 - 6x + 9 + 4 = x^2 - 6x + 13$

(b) $y = (x-3)^2 + 4$

$y - 4 = (x-3)^2$

$\sqrt{y-4} = x - 3$

$\sqrt{y-4} + 3 = x$

$\Rightarrow f^{-1}(x) = \sqrt{x-4} + 3$

(c) $x \geq 3$ and $y \geq 4$

(d) $x \geq 4$ and $y \geq 3$

24.

$y = \dfrac{3x-4}{x+2}$

$xy + 2y = 3x - 4$

simplifying $x(y-3) = -2y - 4$

expressing y in terms of x, $x = \dfrac{2y+4}{3-y}$

interchanging x and y, $y = \dfrac{2x+4}{3-x}$

$f^{-1}(x) = \dfrac{2x+4}{3-x}$

(b) Domain $x \neq 3$, ($x \in \mathbb{R}$ not required)

(c) Since f is the inverse of f^{-1}: $D = \dfrac{3F-4}{F+2}$

GIVEN $f \circ g$ AND ONE OF THE FUNCTIONS FIND THE OTHER ONE

25. (a) $(f \circ g)(x) = 2(5x+3)+5 = 10x+6+5 = 10x+11$

(b) $f^{-1}(x) = \dfrac{x-5}{2}$

Since $g = f^{-1} \circ (f \circ g)$, $g(x) = \dfrac{(10x+11)-5}{2} = \dfrac{10x+6}{2} = 5x+3$

(c) $g^{-1}(x) = \dfrac{x-3}{5}$

Since $f = (f \circ g) \circ g^{-1}$, $f(x) = 10\left(\dfrac{x-3}{5}\right)+11 = 2(x-3)+11 = 2x-6+11 = 2x+5$

26. (a) $f = h \circ g^{-1}$

(b) $g = f^{-1} \circ h$

(c) $g = f^{-1} \circ k \circ h^{-1}$

27. **METHOD A**

$f^{-1}(x) = \sqrt[3]{x}$

(a) Since $g = f^{-1} \circ (f \circ g)$, $g(x) = \sqrt[3]{x+1}$

(b) Since $g = (g \circ f) \circ f^{-1}$, $g(x) = \sqrt[3]{x}+1$

METHOD B

(a) $f(g(x)) = x + 1 \Rightarrow [g(x)]^3 = x + 1$

so $g(x) = \sqrt[3]{x+1}$

(b) $g(f(x)) = x + 1 \Rightarrow g(x^3) = x + 1$

so $g(x) = \sqrt[3]{x}+1$

28. **METHOD A**

$f^{-1}(x) = \sqrt[3]{x+1}$

(a) Since $g = f^{-1} \circ (f \circ g)$, $g(x) = \sqrt[3]{2x+1+1} = \sqrt[3]{2x+2}$

(b) Since $g = (g \circ f) \circ f^{-1}$, $g(x) = 2\sqrt[3]{x+1}+1$

METHOD B

(a) $f(g(x)) = x + 1 \Rightarrow [g(x)]^3 - 1 = 2x + 1$

so $g(x) = \sqrt[3]{2x+2}$

(b) $g(f(x)) = 2x + 1 \Rightarrow g(x^3-1) = 2x + 1$

Set $y = x^3-1$, then $x = \sqrt[3]{y+1}$, so $g(x) = 2\sqrt[3]{x+1}+1$

29.

$g^{-1}(x) = \dfrac{x+1}{2}$

$f(x) = f \circ g \circ g^{-1}(x) = \dfrac{\frac{x+1}{2}+1}{2}$

$= \dfrac{x+3}{4}$

$f(x-3) = \dfrac{(x-3)+3}{4} = \dfrac{x}{4}$

B. Paper 2 questions (LONG)

30. (a) Let $y = \dfrac{x^2-1}{x^2+1} \Rightarrow yx^2 + y = x^2 - 1$

$x^2(1-y) = 1 + y \Rightarrow x^2 = \dfrac{1+y}{1-y} \Rightarrow x = \pm\sqrt{\dfrac{1+y}{1-y}}$ so $f^{-1}(x) = \sqrt{\dfrac{1+x}{1-x}}$

(b) $f^{-1}(x) = -\sqrt{\dfrac{1+x}{1-x}}$

(c) $S = \dfrac{T^2-1}{T^2+1}$

(d) 0.385 (3sf)

(e) – 1

31. Let $f(x) = \sqrt{x+1}+1$ and $g(x) = x^2$.

$(g\circ f)(x) = 1 \Leftrightarrow (\sqrt{x+1}+1)^2 = 1 \Leftrightarrow \sqrt{x+1}+1 = 1 \Leftrightarrow \sqrt{x+1} = 0 \Leftrightarrow x = -1$

For the next two questions we need $f^{-1}(x) = (x-1)^2 - 1 = x^2 - 2x$

(a) $h = g\circ f^{-1}$ so $h(x) = (x^2-2x)^2$

(b) $k = f^{-1}\circ g$ so $k(x) = x^4 - 2x^2$

32. (a) $\dfrac{3x-1}{x-3} = y \Leftrightarrow xy - 3y = 3x - 1 \Leftrightarrow x(y-3) = 3y - 1 \Leftrightarrow x = \dfrac{3y-1}{y-3}$

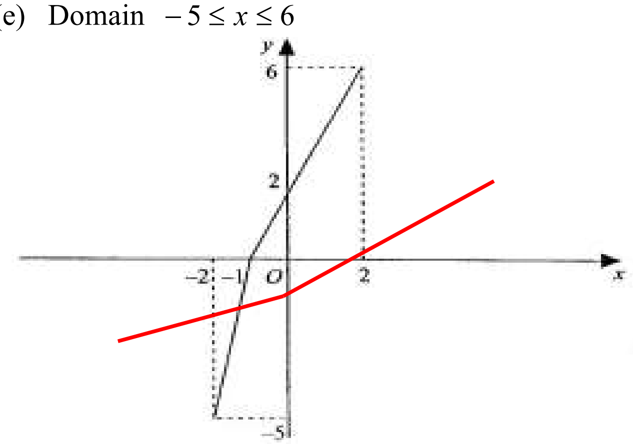
so $f^{-1}(x) = \dfrac{3x-1}{x-3} = f(x)$

(b) $(f\circ f)(k) = k$

(c) $Q = \dfrac{3P-1}{P-3}$

(d) (i) $(f\circ g)(-2) = f(-5) = 2$ (ii) $(g\circ f)(-5) = g(2) = 6$ (iii) $(g\circ g)(0) = g(2) = 6$.

(e) Domain $-5 \le x \le 6$



33. (a) $f^{-1}(x) = (x-1)^2$.

(b) $x \ge 1,\ y \ge 0$

(c) $x = \dfrac{3+\sqrt{5}}{2}$.

(d) $Q = (P-1)^2$, It is the value where P and Q coincide.